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On the Composition and Decomposition of Clutters

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The decomposition theory for simple n-person games is discussed in the context of clutters on a finite set. The notion of committee is defined, and an algorithm is given to find committees of any clutter. It is shown that this decomposition leads to a decomposition scheme for solving bottleneck and shortest path problems over clutters. Finally it is shown that the property of satisfying the length-width inequality is preserved under composition.

INTRODUCTION

In [7], Shapley introduced the notion of a committee of a simple game, and related it to the notion of a component of a compound simple game (see [3], [4], [5], [6] and [7]). He then proved a "unique factorization" theorem for simple games which states that any simple game can be given an essentially unique compound representation involving only prime (i.e., committee-free) games and sums and products of prime games.

In this report we will first rephrase Shapley's definitions and main result in terms of the seemingly more general (but in fact equivalent) combinatorial notion of a clutter on a finite set (see [1] and [2]). In this framework we will present a simple algorithm to discover the existence of committees of a clutter. Such an algorithm is helpful since, if we know that a clutter is not a sum or product, knowledge of one maximal proper committee will enable us to begin the decomposition of the clutter.

Finally, we will discuss non-game applications of the decomposition of clutters. An equivalent version of this decomposition theory has been developed in the context of Boolean functions by Birnbaum and Esary [9]. See also [8].

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COMPOUND CLUTTERS AND COMMITTEES

The notions of a clutter on a finite set and the blocking clutter of a clutter are discussed by Edmonds and Fulkerson [1] and Fulkerson [2]. We will use their terminology and their notation throughout. We begin with some definitions.

By *a family $\mathscr{F}$ on a finite set N* we will mean a family of subsets of N. A family $\mathscr{F}$ on N is said to *cover a point* $i \in N$ if $i \in \bigcup_{F \in \mathscr{F}} F$. $\mathscr{F}$ *covers a subset* $S \subset N$ if $S \subset \bigcup_{F \in \mathscr{F}} F$.

DEFINITION. If $\mathscr{F}$ is a family on N, and if $S \subset N$, then $\mathscr{F}(S) \subset \mathscr{F}$ is a family on N defined by

$$\mathscr{F}(S) = \{F \in \mathscr{F} \mid F \cap S \neq \phi\}.$$

DEFINITION. A family $\mathscr{R}$ on N is called a *clutter* if $\mathscr{R} \neq \phi$, $\mathscr{R} \neq \{\phi\}$ and no element of $\mathscr{R}$ is properly contained in another element of $\mathscr{R}$.

Note. Edmonds and Fulkerson do not exclude the extreme cases ϕ and $\{\phi\}$ from their definition of a clutter. We do so for technical reasons.

DEFINITION. Let $\mathscr{R}$ be a clutter on N. Then the *blocking clutter* of $\mathscr{R}$ (or simply the *blocker* of $\mathscr{R}$), $b[\mathscr{R}]$, is a clutter on N defined by

$$b[\mathscr{R}] = \{S \subset N \mid S \cap R \neq \phi \text{ for all } R \in \mathscr{R}, \text{ and no proper subset of } S \text{ has this property}\}.$$

Edmonds and Fulkerson call the triple $(N, \mathscr{R}, b[\mathscr{R}])$ a blocking system. They have proved that, for any clutter $\mathscr{R}$, $b[b[\mathscr{R}]] = \mathscr{R}$. To find the blocker of any clutter $\mathscr{R}$, one need only list all subsets formed by taking one element from each $R \in \mathscr{R}$, eliminating those which are not minimal.

The concept of a clutter $\mathscr{R}$ on N is equivalent to the concept of an $|N|$-person simple game G in which the elements of $\mathscr{R}$ are considered to be the minimal winning coalitions of G. The clutter $b[\mathscr{R}]$ corresponds to G^*, the dual game of G (see Shapley [3]). An element $i \in N$ which is not covered by $\mathscr{R}$ is known as a dummy player in the game context. Following this equivalence, we now rephrase Shapley's definition of a compound simple game in terms of clutters:

DEFINITION. Let $N_1, \ldots, N_m$ be pairwise disjoint finite sets, and let $\mathscr{R}_i$ be a clutter on N_i, $i = 1, \ldots, m$. Let $\mathscr{R}$ be a clutter on $\{1, \ldots, m\}$. Then we define the *compound clutter on $N = N_1 \cup \cdots \cup N_m$ with quotient $\mathscr{R}$ and components $\mathscr{R}_1, \ldots, \mathscr{R}_m$* to be:

$$\mathscr{R}(\mathscr{R}_1, \ldots, \mathscr{R}_m) = \Big\{\bigcup_{i \in R} R_i \mid R \in \mathscr{R};\ R_i \in \mathscr{R}_i \text{ for all } i \in R\Big\}.$$

In particular, we will distinguish two special compounds, called the sum and product, respectively, which are defined as follows:

$$\mathscr{R}_1 + \cdots + \mathscr{R}_m = \mathscr{R}^+(\mathscr{R}_1, \ldots, \mathscr{R}_m) \quad \text{where} \quad \mathscr{R}^+ = \{\{1\}, \ldots, \{m\}\}.$$

$$\mathscr{R}_1 \times \cdots \times \mathscr{R}_m = \mathscr{R}^\times(\mathscr{R}_1, \ldots, \mathscr{R}_m) \quad \text{where} \quad \mathscr{R}^\times = \{\{1, \ldots, m\}\}.$$

It is easy to see that the sum clutter is merely the union of the sets $\mathscr{R}_i$, while the product clutter is their Cartesian product. Note that $+$ and $\times$ are associative operations.

The following proposition is a restatement of the well-known duality for compound simple games. The proof is omitted.

PROPOSITION. $b[\mathscr{R}(\mathscr{R}_1, \ldots, \mathscr{R}_m)] = b[\mathscr{R}](b[\mathscr{R}_1], \ldots, b[\mathscr{R}_m])$. *In particular, since* $b[\mathscr{R}^+] = \mathscr{R}^\times$ $(b[\mathscr{R}^\times] = \mathscr{R}^+)$, *the blocker of a sum (product) of clutters is the product (sum) of their blocking clutters.*

We say we have a *compound representation* of a clutter if we have written it in some compound form, where the components of this compound may themselves be compound clutters. Every clutter $\mathscr{R}$ can be given two trivial compound representations, namely,

$$\mathscr{R} = \mathscr{B}_1(\mathscr{R}),$$

$$\mathscr{R} = \mathscr{R}(\mathscr{B}_1, \ldots, \mathscr{B}_1),$$

where $\mathscr{B}_1$ is the clutter on a one element set $\{e\}$ which consists of one subset, $\{e\}$ itself. Any other compound representation will be called *non-trivial*. The aim of the present analysis is to discover a non-trivial compound representation of a given clutter, if one exists. To this end we now define a committee of a clutter ([7]).

DEFINITION. Let $\mathscr{R}$ be a clutter on N. A subset $C \subset N$ is called a *committee* of $\mathscr{R}$ if and only if

$$\mathscr{R}(C) = \{(R_1 \cap C) \cup (R_2 - C) \mid R_1, R_2 \in \mathscr{R}(C)\}.$$

If C is a committee of $\mathscr{R}$ then

$$\mathscr{R}_C = \{R \cap C \mid R \in \mathscr{R}(C)\}$$

is a clutter on C, providing that $\mathscr{R}(C) \neq \phi$. Clearly if $\mathscr{R}(C)$ is empty, i.e., no element of C is covered by $\mathscr{R}$, then C is trivially a committee. To eliminate these trivial cases, we will require from now on that $\mathscr{R}$

covers N. This involves no loss of generality, since we are interested in finding a compound representation of a clutter, and elements which are not covered do not enter into consideration.

It is clear from the definition of a compound clutter that, if $\mathscr{R}_k$ is a component of the compound, then the set of elements covered by $\mathscr{R}_k$ (i.e., N_k) is a committee. The converse is also true. That is, if C is a committee of $\mathscr{R}$ then $\mathscr{R}_C$ is a component of some compound representation of $\mathscr{R}$. To see this we make the following definition:

DEFINITION. Let $\mathscr{R}$ be a clutter on N (which covers N). Let $C \subset N$ be a committee of $\mathscr{R}$. We define a new clutter $\mathscr{R}/C$ on the set $(N - C) \cup \{i_C\}$, $i_C \notin N$ a "new" element, called the *contraction of $\mathscr{R}$ on C*, by

$$\mathscr{R}/C = \{R \mid R \in \mathscr{R} - \mathscr{R}(C)\} \cup \{(R - C) \cup \{i_C\} \mid R \in \mathscr{R}(C)\}.$$

Thus $\mathscr{R}/C$ consists of those sets of $\mathscr{R}$ which do not meet C plus those sets of $\mathscr{R}$ which meet C with the elements of C contracted to the one element i_C. $\mathscr{R}/C$ is a clutter since C is a committee. It is obvious that $\mathscr{R}$ can be given the compound representation

$$\mathscr{R} = \mathscr{R}/C(\mathscr{B}_1, \ldots, \mathscr{B}_1, \mathscr{R}_C),$$

where $\mathscr{R}_C$ is in the "slot" operated on by i_C.

The set N and all one element sets are always committees. Any other committees will be called *proper.* A clutter will be called *decomposable* if it has a non-trivial compound representation. By the above discussion it is clear that a clutter is decomposable if and only if it admits a proper committee. All other clutters will be called *prime.*

Suppose any clutter $\mathscr{R}$ can be given a compound representation as either a sum or product or a compound with a prime quotient. The next step would be, since the components of this representation are clutters, to check if they can be further decomposed, and, if so, to write them as sums or products or compounds with prime quotients. This process can be repeated until every component is prime. In this way, we will have produced a compound representation of $\mathscr{R}$ which involves only sums, products, and prime clutters. Shapley's main result is that such a process can be carried out, and the result is unique.

THEOREM (Shapley [7]). *Every clutter has a unique compound representation involving perhaps $\mathscr{R}^+$, $\mathscr{R}^\times$, and otherwise only prime clutters.*

In order to carry out the above procedure, we must, for any clutter $\mathscr{R}$ on N which covers N, be able to determine whether $\mathscr{R}$ is a sum or product (it cannot be both), a prime, or a compound other than sum or product.

$\mathscr{R}$ is a sum if and only if N can be partitioned into $N_1, N_2, \ldots, N_m$ such that each $R \in \mathscr{R}$ is a subset of some N_k. In this case

$$\mathscr{R} = \mathscr{R}_{N_1} + \mathscr{R}_{N_2} + \cdots + \mathscr{R}_{N_m}.$$

We choose the finest such partition guaranteeing that none of the $\mathscr{R}_{N_k}$ is itself a sum.

$\mathscr{R}$ is a product if and only if $b[\mathscr{R}]$ is a sum. In this case we decompose $b[\mathscr{R}]$ as a sum as above, and the "factors" of $\mathscr{R}$ will be the blockers of the "summands" of $b[\mathscr{R}]$.

If $\mathscr{R}$ is neither sum nor product, then Shapley proved that the maximal proper committees of $\mathscr{R}$ must be disjoint. If there are no proper committees, then $\mathscr{R}$ is prime. Otherwise if $C_1, \ldots, C_m$ are the maximal proper committees of $\mathscr{R}$ in any order, then

$$\mathscr{S} = \mathscr{R}/C_1/C_2/ \cdots /C_m$$

will be prime, and $\mathscr{R}$ is of the form

$$\mathscr{R} = \mathscr{S}(\mathscr{R}_{C_1}, \mathscr{R}_{C_2}, \ldots, \mathscr{R}_{C_m}, \mathscr{B}_1, \ldots, \mathscr{B}_1).$$

Shapley also proved that the above multiple contraction can be carried out. That is, he proved that, if C is a committee of $\mathscr{R}$, and $D \subseteq N - C$, then D is a committee of $\mathscr{R}$ if and only if D is a committee of $\mathscr{R}/C$.

Thus the problem remaining is, given a clutter $\mathscr{R}$ which is neither sum nor product, to determine if $\mathscr{R}$ admits a proper committee, and, if so, to find a maximal proper committee. If we contract on this committee and repeat the process, we will eventually reach the clutter $\mathscr{S}$ discussed above.

In the next section we will describe an algorithm to find a maximal proper committee of a clutter, if one exists.

Before moving on, we would like to repeat that nothing in this section is new. All the ideas and results are due to Shapley; we have merely renamed them. We feel that the renaming will simplify the description of the algorithm, and we hope that it will facilitate applications of these results to combinatorial problems of a non-game-theoretic nature.

The Algorithm

Suppose $\mathscr{R}$ is a clutter on N which covers N. We will describe an algorithm which for any subset $T \subset N$ will give the smallest committee C_T of $\mathscr{R}$ containing T. That is, if C' is a committee of $\mathscr{R}$, $C' \supset T$ implies $C' \supset C_T$. We note that, for any T, there always exists a unique smallest committee C_T containing T since by [7, Theorem 7] the non-empty intersection of two committees is a committee. However C_T need not be proper.

Before giving the algorithm, we show it solves the decomposition problem stated in the last section. Suppose $\mathscr{R}$ is neither a sum nor a product. We want to find a maximal proper committee of $\mathscr{R}$, if one exists. Let $N = \{1, 2,..., n\}$. Let $T_{ij} = \{i, j\}$, for $i = 1,..., n$; $j = i,..., n$. If for each of the $\binom{n}{2}$ T_{ij} defined, $C_{T_{ij}} = N$, then $\mathscr{R}$ is prime. If not, then for some pair (k, l), $k < l$, $C_{T_{kl}} \equiv C^{(1)}$ is a proper committee. Suppose we have chosen the "first" pair (k, l) such that this happens. That is, no point $i < k$ is in a proper committee; and no pair (k, j) $j < l$ is in a proper committee.

Now let $C_i^{(1)} = C^{(1)} \cup \{i\}$ for $i \in \{j \mid j \notin C^{(1)}, j > l\}$. If, for each of the $C_i^{(1)}$ defined, $C_{C_i^{(1)}} = N$, then $C^{(1)}$ is a maximal proper committee. This follows by the choice of k and l. Otherwise let ν be the smallest index i for which $C_{C_i^{(1)}}$ is proper, and let $C^{(2)} = C_{C_\nu^{(1)}}$. Again define

$$C_i^{(2)} = C^{(2)} \cup \{i\} \qquad \text{for} \quad i \in \{j \mid j \notin C^{(2)}, j > \nu\}$$

and continue. This process will eventually terminate at a maximal proper committee $C^{(\alpha)}$. In fact we will have

$$\{k, l\} \subseteq C^{(1)} \subset C^{(2)} \subset \cdots \subset C^{(\alpha)} \subset N$$

with all the inclusions strict, except possibly the first.

Briefly, what we have just done is the following. First we found a proper committee $C^{(1)}$ containing some pair of elements of N. (If no pair is in a proper committee, the clutter is prime.) We then perturbed that committee by adding a new element. If, for every element we add, the smallest committee containing the enlarged set is N, then $C^{(1)}$ is maximal. Otherwise we obtain a larger committee $C^{(2)}$, which we perturb to check for maximality, and so on.

We now return to the problem of given T to find C_T. Let $\mathscr{R}$ be a clutter on N which covers N. For each $i \in N$, let

$$\mathscr{R}^i = \{R - \{i\} \mid R \in \mathscr{R}(\{i\})\}.$$

Since $\mathscr{R}$ covers N, $\mathscr{R}^i \neq \phi$ for $i \in N$. Thus, if $\mathscr{R}^i \neq \{\phi\}$, then $\mathscr{R}^i$ is a clutter on $N - \{i\}$.

Since $\mathscr{R}^k = \{\phi\}$ implies

$$\mathscr{R} = \{\{k\}\} + \mathscr{R}(N - \{k\})$$

we could also eliminate that non-clutter possibility. However we prefer to keep it and treat it independently. We define $b[\{\phi\}] \equiv \phi$ and $b[\phi] \equiv \{\phi\}$.

For each $i \in N$, let $\mathscr{S}^i = b[\mathscr{R}^i]$.

Let $T \subset N$ be given. We want to find C_T, the smallest committee containing T. Let $T_1 = T$.

Proceeding inductively, suppose T_k has been defined. Let

$$\mathscr{S}_k = \bigcup_{j \in T_k} \mathscr{S}^j - \bigcap_{j \in T_k} \mathscr{S}^j,$$

a family (not necessarily a clutter). If $\mathscr{S}_k$ covers no $i \in N - T_k$, then $C = T_k$ is the required committee. If not, define

$$T_{k+1} = \bigcup \{S \mid S \in \mathscr{S}_k \cup \{T_k\}\}$$

and continue. Since $T_{k+1} \supset T_k$ strictly, this process will eventually terminate (possibly with $T_l = N = C$).

THEOREM 1. $C = C_T$, *the smallest committee containing* T.

Proof. We show first that C is a committee, then that it is the smallest. Obviously $C \supset T$.

Since sums and products have been defined only for clutters, we make the following special definitions. Let $\mathscr{B}$ be a family. Then we define:

$$\mathscr{B} + \phi \equiv \mathscr{B},$$

$$\mathscr{B} \times \{\phi\} \equiv \mathscr{B}.$$

We note that, for $i \in N$,

$$\mathscr{R}(\{i\}) = \{\{i\}\} \times \mathscr{R}^i.$$

Since the process terminated with $C = T_k$ for some k, we have

$$\mathscr{S}_k = \bigcup_{j \in C} \mathscr{S}^j - \bigcap_{j \in C} \mathscr{S}^j$$

covers no $i \in N - C$. Since, for $i \in N$, $\mathscr{R}^i$ does not cover i, we have $\bigcap_{j \in C} \mathscr{S}^j$ covers no $i \in C$. Hence, for all $i \in C$, we have

$$\mathscr{S}^i = \Big(\mathscr{S}^i - \bigcap_{j \in C} \mathscr{S}^j\Big) + \bigcap_{j \in C} \mathscr{S}^j,$$

where the first is a clutter on C (or is ϕ), the second a clutter on $N - C$ (or is ϕ).

Thus, for all $i \in C$,

$$\mathscr{R}^i = \mathscr{F}^i \times \mathscr{H},$$

where $\mathscr{F}^i$ is a clutter on C (or is $\{\phi\}$) and $\mathscr{H}$ is a clutter on $N - C$ (or is $\{\phi\}$). Therefore, for $i \in C$,

$$\begin{aligned} \mathscr{R}(\{i\}) &= \{\{i\}\} \times \mathscr{F}^i \times \mathscr{H} \\ &= \mathscr{G}^i \times \mathscr{H}, \end{aligned}$$

with $\mathscr{G}^i$ a clutter on C.

In particular, we must have, for all $i \in C$,

$$\begin{aligned} \mathscr{G}^i &= \{R \cap C \mid R \in \mathscr{R}(\{i\})\}, \\ \mathscr{H} &= \{R - C \mid R \in \mathscr{R}(\{i\})\}. \end{aligned}$$

But, since $\mathscr{R}(C) = \bigcup_{i \in C} \mathscr{R}(\{i\})$, we have in fact

$$\mathscr{H} = \{R - C \mid R \in \mathscr{R}(C)\}.$$

It is obvious that

$$\mathscr{R}(C) \subset \{(R_1 \cap C) \cup (R_2 - C) \mid R_1, R_2 \in \mathscr{R}(C)\}.$$

To show equality, suppose

$$R = (R_1 \cap C) \cup (R_2 - C)$$

for some $R_1, R_2 \in \mathscr{R}(C)$. Then $R_1 \in \mathscr{R}(\{i\})$ for some $i \in C$, and thus, for that i,

$$R \in \mathscr{G}^i \times \mathscr{H} = \mathscr{R}(\{i\}) \subset \mathscr{R}(C).$$

This proves C is a committee.

To show C is in fact the smallest, consider C_T. Since C_T is a committee,

$$\mathscr{R}(C_T) = \{(R_1 \cap C_T) \cup (R_2 - C_T) \mid R_1, R_2 \in \mathscr{R}(C_T)\}.$$

Hence, for $i \in C_T$,

$$\mathscr{R}^i = \mathscr{U}^i \times \mathscr{V},$$

where

$$\begin{aligned} \mathscr{U}^i &= \{R \cap (C_T - \{i\}) \mid R \in \mathscr{R}(\{i\})\}, \\ \mathscr{V} &= \{R - C_T \mid R \in \mathscr{R}(C_T)\}, \end{aligned}$$

are clutters (or $\{\phi\}$) on C_T and $N - C_T$, respectively. Thus, for $i \in C_T$,

$$\mathscr{S}^i = b[\mathscr{U}^i] + b[\mathscr{V}],$$

with $b[\mathscr{U}^i]$ and $b[\mathscr{V}]$ clutters (or ϕ) on C_T and $N - C_T$, respectively.

Suppose, for some l, $T \subsetneqq T_l \subsetneqq C_T$. Then

$$T_{l+1} = \bigcup \Big\{S \mid S \in \Big(\bigcup_{j \in T_l} \mathscr{S}^j - \bigcap_{j \in T_l} \mathscr{S}^j\Big) \cup \{T_l\}\Big\}.$$

Since

$$\bigcap_{j \in T_l} \mathscr{S}^j \supset b[\mathscr{V}],$$

we must have

$$\bigcup_{j \in T_l} \mathscr{S}^j - \bigcap_{j \in T_l} \mathscr{S}^j \subset \bigcup_{j \in T_l} b[\mathscr{U}^i],$$

and thus the family on the left of the inclusion is a family on C_T. Thus we have $T_{l+1} \subsetneqq C_T$, which implies $C \subsetneqq C_T$, yielding finally $C = C_T$. This completes the proof of the theorem.

Applications

Let $N_1, ..., N_m$ be pairwise disjoint finite sets, and let $\mathscr{R}_i$ be a clutter on N_i, $i = 1, ..., m$. Let $\mathscr{R}$ be a clutter on $E = \{1, ..., m\}$ and define $\mathscr{P} = \mathscr{R}(\mathscr{R}_1, ..., \mathscr{R}_m)$, a clutter on $N = N_1 \cup \cdots \cup N_m$.

Proposition 1. *For any real valued function f on N,*

$$\min_{P \in \mathscr{P}} \max_{x \in P} f(x) = \min_{R \in \mathscr{R}} \max_{e \in R} f'(e),$$

where f' is a real valued function defined on E by

$$f'(e) = \min_{R \in \mathscr{R}_e} \max_{x \in R} f(x).$$

Proof. In [1] and [2] it is shown that, if the elements x of N are ordered in increasing order of $f(x)$, then $\min_{P \in \mathscr{P}} \max_{x \in P} f(x)$ is attained at a point $x_0 \in N$ such that

$$P(x_0) = \{x \in N \mid f(x) \leqslant f(x_0)\}$$

first contains some $P \in \mathscr{P}$.

Let $x_0 \in N_{e_0}$. Then $f(x_0) = f'(e_0)$. This follows by the fact that we have, in particular, ordered the elements of N_{e_0} in increasing order of $f(x)$, and x_0 must be such that $P(x_0)$ first contains some $R \in \mathscr{R}_{e_0}$. A similar argument on the set E shows

$$f'(e_0) = \min_{R \in \mathscr{R}} \max_{e \in R} f'(e),$$

which completes the proof.

Notice that, by substituting $-f$ for f, we get a version of Proposition 1 for problems of the form $\max_{P\in\mathscr{P}} \min_{x\in P} f(x)$. Both the max min and the min max problems are known as the bottleneck problems. What Proposition 1 says is that to solve a bottleneck problem on a compound clutter one may first solve the problem on each of the components and then, with these results as inputs, solve the problem on the quotient. Thus decomposition of the clutter leads to a decomposition scheme for the bottleneck problems.

PROPOSITION 2. *For any real valued function f on N,*

$$\min_{P\in\mathscr{P}} \sum_{x\in P} f(x) = \min_{R\in\mathscr{R}} \sum_{e\in R} f'(e),$$

where f' is a real valued function defined on E by

$$f'(e) = \min_{R\in\mathscr{R}_e} \sum_{x\in R} f(x).$$

Proof. This follows from

$$\min_{P\in\mathscr{P}} \sum_{x\in P} f(x) = \min_{R\in\mathscr{R}} \min_{R_e\in\mathscr{R}_e} \sum_{e\in R} \sum_{x\in R_e} f(x)$$

$$= \min_{R\in\mathscr{R}} \sum_{e\in R} \min_{R_e\in\mathscr{R}_e} \sum_{x\in R_e} f(x).$$

As before, by substituting $-f$ for f, we get a version of Proposition 2 for $\max_{P\in\mathscr{P}} \sum_{x\in P} f(x)$.

Let $\mathscr{P}$ be a clutter on N. After Fulkerson [2], we call the triple $(N, \mathscr{P}, b[\mathscr{P}])$ a blocking system. The following definition is due to Fulkerson.

DEFINITION. *The length-width inequality* holds for $(N, \mathscr{P}, b[\mathscr{P}])$ if

$$\Big(\min_{P\in\mathscr{P}} \sum_{x\in P} l(x)\Big)\Big(\min_{K\in b[\mathscr{P}]} \sum_{x\in K} w(x)\Big) \leqslant \sum_{x\in N} l(x)\, w(x)$$

is satisfied for every pair of non-negative real valued functions l and w defined on N.

It is easy to see that, if $M \supset N$, then the length-width inequality holds for $(N, \mathscr{P}, b[\mathscr{P}])$ if and only if it holds for $(M, \mathscr{P}, b[\mathscr{P}])$. Thus with no loss of generality we may say the length-width inequality holds for a clutter $\mathscr{P}$ if it holds for $(\bigcup_{P\in\mathscr{P}} P, \mathscr{P}, b[\mathscr{P}])$. In other words, we need only consider clutters $\mathscr{P}$ on N where $\mathscr{P}$ covers N.

THEOREM 2. *Let* $\mathscr{P} = \mathscr{R}(\mathscr{R}_1, \ldots, \mathscr{R}_m)$ *be a compound clutter on* $N = N_1 \cup \cdots \cup N_m$ *as defined earlier. If the length-width inequality holds for* $\mathscr{R}$ *and for each of* $\mathscr{R}_1, \ldots, \mathscr{R}_m$, *then it holds for* $\mathscr{P}$.

Proof. Suppose l and w are non-negative real valued functions defined on N. For each $e \in E = \{1, \ldots, m\}$ define

$$l'(e) = \min_{R \in \mathscr{R}_e} \sum_{x \in R} l(x),$$

$$w'(e) = \min_{K \in b[\mathscr{R}_e]} \sum_{x \in K} w(x).$$

Let

$$\lambda = \min_{P \in \mathscr{P}} \sum_{x \in P} l(x) \quad \text{and} \quad \omega = \min_{K \in b[\mathscr{P}]} \sum_{x \in K} w(x).$$

By Proposition 2, we have

$$\lambda = \min_{R \in \mathscr{R}} \sum_{e \in R} l'(e), \quad \text{and} \quad \omega = \min_{K \in b[\mathscr{R}]} \sum_{e \in K} w'(e).$$

Since the length-width inequality holds for $\mathscr{R}$, we have

$$\lambda\omega \leqslant \sum_{e \in E} l'(e)\, w'(e).$$

For each $e \in E$, the length-width inequality holds for $\mathscr{R}_e$, thus

$$\begin{aligned} l'(e)\, w'(e) &= \Big(\min_{R \in \mathscr{R}_e} \sum_{x \in R} l(x)\Big)\Big(\min_{K \in b[\mathscr{R}_e]} \sum_{x \in K} w(x)\Big) \\ &\leqslant \sum_{x \in N_e} l(x)\, w(x). \end{aligned}$$

Thus

$$\lambda\omega \leqslant \sum_{e \in E} \sum_{x \in N_e} l(x)\, w(x) = \sum_{x \in N} l(x)\, w(x),$$

which completes the proof.

Given clutters for which the length-width inequality holds (see [2]), Theorem 2 enables us to generate additional clutters having this property. One open question is whether any new clutters can be discovered this way. It is easy to check, for example, if the components and quotient are clutters arising from paths (from sources to sinks) in graphs; then the compound arises as the paths in a suitable compounding of the graphs (for each edge of the "quotient" graph substitute the corresponding "component" graph, identifying the source with the vertex on one end, and the sink with the vertex on the other end).

In [10], Bixby has proved that the converse of Theorem 2 holds. That is, if the length-width inequality holds for $\mathscr{P}$, and $\mathscr{P} = \mathscr{R}(\mathscr{R}_1, \ldots, \mathscr{R}_m)$, then the length-width inequality holds for $\mathscr{R}$ and $\mathscr{D}_1, \ldots, \mathscr{R}_m$.

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